

Ising Model and \mathbf{Z}_2 Electrodynamics

Yury M. Zinoviev*

Steklov Mathematical Institute, Gubkin Street 8, 119991, Moscow, Russia,
e - mail: zinoviev@mi.ras.ru

Abstract. The correlation functions and spontaneous magnetization are calculated for the three-dimensional Ising model and for the three-dimensional \mathbf{Z}_2 electrodynamics.

1 Introduction

We consider first the two-dimensional Ising model for a square lattice of M columns and N rows. The Hamiltonian is

$$\overline{H}(\sigma_{11}, \dots, \sigma_{NM}) = -H \sum_{m,n} \sigma_{nm} - J_1 \sum_{m,n} \sigma_{nm} \sigma_{n+1,m} - J_2 \sum_{m,n} \sigma_{nm} \sigma_{n,m+1}. \quad (1.1)$$

σ_{nm} is a classical variable taking on the values ± 1 . For boundary conditions we can assume either that the lattice is wrapped on a torus or we can assume that the lattice has free ends. For an $M \times N$ lattice the free energy per site and the magnetic moment per site (magnetization) are defined by

$$F_{MN}(H) = \frac{1}{\beta MN} \ln \left(\sum_{\sigma_{nm} = \pm 1} \exp\{-\beta \overline{H}(\sigma_{11}, \dots, \sigma_{NM})\} \right),$$

$$M_{MN}(H) = \frac{1}{MN} \frac{\sum_{\sigma_{nm} = \pm 1} \sum_{m,n} \sigma_{mn} \exp\{-\beta \overline{H}(\sigma_{11}, \dots, \sigma_{NM})\}}{\sum_{\sigma_{nm} = \pm 1} \exp\{-\beta \overline{H}(\sigma_{11}, \dots, \sigma_{NM})\}} = \frac{\partial F_{MN}}{\partial H}. \quad (1.2)$$

The constant $\beta = (kT)^{-1}$ is positive. Yang [1] calculated the spontaneous magnetization

$$M_{Yang} = \left| \lim_{\alpha \rightarrow 0+} \lim_{M, N \rightarrow \infty} M_{MN}(\alpha/M) \right|. \quad (1.3)$$

M and N tend to infinity together, i.e., with M/N a fixed ratio. The spontaneous magnetization of Montroll, Potts and Ward [2] is

$$M_{MPW}^2 = \lim_{M, N \rightarrow \infty} (MN)^{-2} \sum_{n, m, n', m'} < \sigma_{n'm'}^x \sigma_{nm}^x >_{MN}, \quad (1.4)$$

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where $\langle \dots \rangle_{MN}$ denotes a thermal average in zero field for an $M \times N$ lattice. In order to describe the situation in the two-dimensional Ising model we cite the paper [3]:

"Although a great deal of effort has been spent on the two-dimensional Ising model, the amount of exact results is remarkably limited. For the case of the rectangular lattice without magnetic field Onsager and Kaufman [4] - [6] have given the free energy per lattice site and also the correlation functions for spins at finite distances. In particular, it is readily observed that the expression for the two-spin correlation function becomes rapidly more and more complicated as the separation between the two spins increases. It is for this reason that it is quite difficult to obtain, as first accomplished by Yang [1], the spontaneous magnetization, which is closely related to the limiting value at infinite separations of the two-spin correlation function."

Schultz, Mattis and Lieb [7] write on the paper of Yang [1] and on the paper of Montroll, Potts and Ward [2]:

"In contrast to the free energy, the spontaneous magnetization of the Ising model on a square lattice, correctly defined, has never been solved with complete mathematical rigor. Starting from the only sensible definition of the spontaneous magnetization, the methods of Yang, and of Montroll, Potts, and Ward are each forced to make an assumption that has not been rigorously justified. The assumptions appear to be quite different; however, from the similarities between the difficulties encountered in trying to justify them, and the identity of the results obtained, one might conclude that they are closely related."

Isakov [8] obtained the estimates of the derivatives of the magnetization with respect to the magnetic field for zero magnetic field and sufficiently large βJ_1 and βJ_2 . These estimates imply that the magnetization cannot be a holomorphic function of the magnetic field.

In the paper [9] the new definition of the spontaneous magnetization was suggested by making use of the one-dimensional Ising model. Let the number $\sigma_k = \pm 1$, $k = 1, \dots, N+1$, $\sigma_{N+1} = \sigma_1$, be given. The partition function of the Ising model with the constant magnetic field $H(k) = H$, $k = 1, \dots, N$, is

$$Z_0(J, H; T(1, N)) = \sum_{\substack{\sigma_k = \pm 1, k=1, \dots, N+1, \\ \sigma_{N+1} = \sigma_1}} \exp\{\beta J \sum_{k=1}^N \sigma_k \sigma_{k+1} + \beta H \sum_{k=1}^N \sigma_k\}. \quad (1.5)$$

Due to ([10], Chapter II, formula (4.5), Chapter III, formulas (2.10), (2.13)) the partition function and the average total magnetization are

$$\begin{aligned} Z_0(J, H; T(1, N)) &= (\lambda_+(J, H))^N + (\lambda_-(J, H))^N, \\ \lambda_{\pm}(J, H) &= \exp\{\beta J\}(\cosh \beta H \pm (\sinh^2 \beta H + \exp\{-4\beta J\})^{1/2}), \end{aligned} \quad (1.6)$$

$$\overline{M}_0(J, H; T(1, N)) = \beta^{-1} \frac{\partial}{\partial H} \ln Z_0(J, H; T(1, N)), \quad (1.7)$$

$$\lim_{N \rightarrow \infty} N^{-1} \overline{M}_0(J, H; T(1, N)) = (\sinh^2 \beta H + \exp\{-4\beta J\})^{-1/2} \sinh \beta H. \quad (1.8)$$

For the vacuum ($J = 0$) the definition (1.5) implies

$$Z_0(0, H; T(1, N)) = (2 \cosh \beta H)^N \quad (1.9)$$

and the magnetization (1.8) for $J = 0$ is equal to $\tanh \beta H$. It seems natural that the magnetization (1.8) (magnetic moment per site) of the vacuum ($J = 0$) should be zero.

One edge has two boundary vertices. The "energy" of the magnetic field for an edge is the product of the magnetic fields corresponding to the boundary vertices of edge. Summing up the "energies" of the magnetic field $H(k) = H, k = 1, \dots, N$, over all edges we get

$$\sum_{k=1}^N H(k)H(k+1) = NH^2, \quad H(N+1) = H(1). \quad (1.10)$$

The average total magnetization (1.7) and the "energy" (1.10) become infinite for $N \rightarrow \infty$. In order to obtain the finite values in the quantum field theory the "re-normalized" constants are used. In the definition (1.3) Yang used the "re-normalized" constant magnetic field $H/M = (M/N)^{-1/2}(MN)^{-1/2}H$. The lattice has M columns and N rows with the fixed ratio M/N . We consider the "re-normalized" constant magnetic field

$$H(k) = N^{-1/2}\beta^{-1} \tanh \beta H, \quad k = 1, \dots, N, \quad (1.11)$$

to get the finite "energy" (1.10) for $N \rightarrow \infty$. In view of the relations (1.6), (1.9) we get the spontaneous magnetization for the "re-normalized" constant magnetic field (1.11)

$$\lim_{N \rightarrow \infty} \frac{\partial}{\partial x} \left(\ln \frac{Z_0(J, N^{-1/2}\beta^{-1}x; T(1, N))}{Z_0(0, N^{-1/2}\beta^{-1}x; T(1, N))} \right)_{x = \tanh \beta H} = (\exp\{2\beta J\} - 1) \tanh \beta H. \quad (1.12)$$

For the vacuum ($J = 0$) the spontaneous magnetization (1.12) is equal to zero. The value (1.12) is called the spontaneous magnetization since the "re-normalized" constant magnetic field (1.11) tends to zero when $N \rightarrow \infty$.

Due to ([10], Chapter III, formula (3.1)) the two-spin correlation function

$$\langle \sigma_m \sigma_n \rangle_N = (Z_0(J, 0; T(1, N)))^{-1} \sum_{\substack{\sigma_k = \pm 1, \quad k = 1, \dots, N+1, \\ \sigma_{N+1} = \sigma_1}} \sigma_m \sigma_n \exp\{\beta J \sum_{k=1}^N \sigma_k \sigma_{k+1}\}, \quad (1.13)$$

$m, n = 1, \dots, N$. The definitions (1.5), (1.13) and the relation (1.9) imply

$$\sum_{m, n = 1, \dots, N, \quad m \neq n} \langle \sigma_m \sigma_n \rangle_N = \beta^{-2} \frac{\partial^2}{\partial H^2} \left(\ln \frac{Z_0(J, H; T(1, N))}{Z_0(0, H; T(1, N))} \right)_{H=0}. \quad (1.14)$$

The relations (1.6), (1.9), (1.14) imply

$$\lim_{N \rightarrow \infty} N^{-1} \sum_{m, n = 1, \dots, N, \quad m \neq n} \langle \sigma_m \sigma_n \rangle_N = \exp\{2\beta J\} - 1. \quad (1.15)$$

By making use of the relation (1.15) it is possible to express the spontaneous magnetization (1.12) through the two-spin correlation functions

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\partial}{\partial x} \left(\ln \frac{Z_0(J, N^{-1/2}\beta^{-1}x; T(1, N))}{Z_0(0, N^{-1/2}\beta^{-1}x; T(1, N))} \right)_{x = \tanh \beta H} = \\ (2 \tanh \beta H) \lim_{N \rightarrow \infty} N^{-1} \sum_{m, n = 1, \dots, N, \quad m < n} \langle \sigma_m \sigma_n \rangle_N. \end{aligned} \quad (1.16)$$

The magnetization (1.8) expression is cumbersome. For the Ising model (1.5) the right-hand side of the equality of the type (1.4) is equal to zero. For the sufficiently small βJ_1 and βJ_2 the equality of the type (1.16) is proved for the two-dimensional Ising model in the paper [9].

In this paper the correlation functions are calculated and the equality of the type (1.16) is proved for the d -dimensional Ising model ($d = 1, 2, 3$) with the free boundary conditions and for the d -dimensional \mathbf{Z}_2 electrodynamics ($d = 2, 3$) with the free boundary conditions. \mathbf{Z}_2 electrodynamics was introduced in the paper [11].

2 Correlation Functions

Let us define Ising model and \mathbf{Z}_2 electrodynamics by making use of the algebraic topology notations. We consider a rectangular lattice formed by the points with integral Cartesian coordinates $x_i = k_i$, $M'_i \leq k_i \leq M_i$, $i = 1, \dots, d$, $1 \leq d \leq 3$, and the corresponding edges connecting these vertices. We denote this graph by $G(M'_1, \dots, M'_d; M_1, \dots, M_d)$ or simply $G(M)$. We consider the free boundary conditions. For the periodic boundary conditions the lattice is wrapped on a torus. The graph $G(M'_1, \dots, M'_d; M_1, \dots, M_d)$ cells: vertices, edges, faces ($d = 2, 3$), cubes ($d = 3$) are called the cells of dimension 0, 1, 2, 3. They are denoted by $s_i^0, s_i^1, s_i^2, s_i^3$. The cell complex $P(G(M))$ consists of the vertices of the graph $G(M'_1, \dots, M'_d; M_1, \dots, M_d)$ and of the cells of dimension 1, 2, 3 whose boundaries contain the cells of the graph $G(M'_1, \dots, M'_d; M_1, \dots, M_d)$. Let $\mathbf{Z}_2^{add} = \{0, 1\}$ be the group of modulo 2 residuals. The modulo 2 residuals are multiplied by each other and the group \mathbf{Z}_2^{add} is the field. To every pair of the cells s_i^p, s_j^{p-1} there corresponds the incidence number $(s_i^p : s_j^{p-1}) \in \mathbf{Z}_2^{add}$. If the cell s_j^{p-1} is included into the boundary of the cell s_i^p , then the incidence number $(s_i^p : s_j^{p-1}) = 1 \in \mathbf{Z}_2^{add}$. Otherwise the incidence number $(s_i^p : s_j^{p-1}) = 0 \in \mathbf{Z}_2^{add}$. For any pair of the cells s_i^{p+1}, s_j^{p-1} the incidence numbers satisfy the condition

$$\sum_{s_m^p \in P(G(M))} (s_i^{p+1} : s_m^p)(s_m^p : s_j^{p-1}) = 0. \quad (2.1)$$

A cochain c^p of the cell complex $P(G(M))$ with the coefficients in the group \mathbf{Z}_2^{add} is a function on the p -dimensional cells taking values in the group \mathbf{Z}_2^{add} . Usually the oriented cells $\pm s^p$ are considered and the cochains are the antisymmetric functions: $c^p(-s^p) = -c^p(+s^p)$. However, $-1 = 1 \pmod{2}$ and we can neglect the cell orientation for the coefficients in the group \mathbf{Z}_2^{add} : $c^p(-s^p) = c^p(+s^p)$. The cochains form an Abelian group $C^p(P(G(M)), \mathbf{Z}_2^{add})$

$$(c^p + c'^p)(s_i^p) = c^p(s_i^p) + c'^p(s_i^p). \quad (2.2)$$

The homomorphism

$$\partial c^p(s_i^{p-1}) = \sum_{s_j^p \in P(G(M))} (s_j^p : s_i^{p-1}) c^p(s_j^p) \quad (2.3)$$

of the group $C^p(P(G(M)), \mathbf{Z}_2^{add})$ into the group $C^{p-1}(P(G(M)), \mathbf{Z}_2^{add})$ is called the boundary operator. Let us introduce the bilinear form on the group $C^p(P(G(M)), \mathbf{Z}_2^{add})$:

$$\langle f^p, g^p \rangle = \sum_{s_i^p \in P(G(M))} f^p(s_i^p) g^p(s_i^p). \quad (2.4)$$

The homomorphism

$$\partial^* c^p(s_i^{p+1}) = \sum_{s_j^p \in P(G(M))} (s_i^{p+1} : s_j^p) c^p(s_j^p) \quad (2.5)$$

of the group $C^p(P(G(M)), \mathbf{Z}_2^{add})$ into the group $C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$ is called the coboundary operator:

$$\langle f^p, \partial^* g^{p-1} \rangle = \langle \partial f^p, g^{p-1} \rangle, \quad \langle f^p, \partial g^{p+1} \rangle = \langle \partial^* f^p, g^{p+1} \rangle. \quad (2.6)$$

The kernel $Z_p(P(G(M)), \mathbf{Z}_2^{add})$ of the homomorphism (2.3) is called the group of cycles of the complex $P(G(M))$ with the coefficients in the group \mathbf{Z}_2^{add} . The image $B_{p-1}(P(G(M)), \mathbf{Z}_2^{add})$ of the homomorphism (2.3) is called the group of boundaries of the complex $P(G(M))$ with

the coefficients in the group \mathbf{Z}_2^{add} . The condition (2.1) implies $\partial^* \partial^* = 0$ and $\partial \partial = 0$: the group $B_{p-1}(P(G(M)), \mathbf{Z}_2^{add})$ is the subgroup of the group $Z_{p-1}(P(G(M)), \mathbf{Z}_2^{add})$.

The energy is the function on the cochains $\sigma^p \in C^p(P(G(M)), \mathbf{Z}_2^{add})$

$$\overline{H}'_0(\partial^* \sigma^p) = \sum_{s_i^{p+1} \in P(G(M))} h(\partial^* \sigma^p(s_i^{p+1}); s_i^{p+1}) \quad (2.7)$$

where an arbitrary function $h(\epsilon; s_i^{p+1})$ on \mathbf{Z}_2^{add} depends on the cell s_i^{p+1} :

$$h(\epsilon; s_i^{p+1}) = D(s_i^{p+1}) - J(s_i^{p+1})(-1)^\epsilon, \quad (2.8)$$

$$D(s_i^{p+1}) = \frac{1}{2} \left(h(1; s_i^{p+1}) + h(0; s_i^{p+1}) \right), \quad J(s_i^{p+1}) = \frac{1}{2} \left(h(1; s_i^{p+1}) - h(0; s_i^{p+1}) \right).$$

$\epsilon \rightarrow (-1)^\epsilon$ is the isomorphism of the additive group $\mathbf{Z}_2^{add} = \{0, 1\}$ into the multiplicative group $\mathbf{Z}_2 = \{\pm 1\}$. The substitution of the equality (2.8) into the equality (2.7) gives

$$\overline{H}'_0(\partial^* \sigma^p) = D + \overline{H}_0(\partial^* \sigma^p), \quad D = \sum_{s_i^{p+1} \in P(G(M))} D(s_i^{p+1}), \quad (2.9)$$

$$\overline{H}_0(\partial^* \sigma^p) = - \sum_{s_i^{p+1} \in P(G(M))} J(s_i^{p+1})(-1)^{\partial^* \sigma^p(s_i^{p+1})}. \quad (2.10)$$

The number $\partial^* \sigma^p(s_i^{p+1})$ is given by the relation (2.5). Let the interaction energy $J(s_i^1)$ depend on the edge s_i^1 orientation only: J_1 (J_2) is $J(s_i^1)$ for the horizontally (vertically) oriented edges s_i^1 . For $d = 2$, $p = 0$ and for the numbers $\sigma_{n,m} = (-1)^{\sigma^0(s_{n,m}^0)} = \pm 1$ the function (2.10) is the energy (1.1), $H = 0$, for the two-dimensional Ising model

$$-J_1 \sum_{n=M'_1-1}^{M_1} \sum_{m=M'_2}^{M_2} \sigma_{n,m} \sigma_{n+1,m} - J_2 \sum_{n=M'_1}^{M_1} \sum_{m=M'_2-1}^{M_2} \sigma_{n,m} \sigma_{n,m+1}. \quad (2.11)$$

There are no vertices $(M'_1 - 1, m)$, $(n, M'_2 - 1)$, $(M_1 + 1, m)$, $(n, M_2 + 1)$ in the cell complex $P(G(M))$. The values $\sigma_{M'_1-1,m}$, σ_{n,M'_2-1} , $\sigma_{M_1+1,m}$, σ_{n,M_2+1} are equal to one. The term $-J_1 \sigma_{M'_1-1,m}$ in the sum (2.11) and other boundary terms are neglected in the Hamiltonian (1.1). For $p = 1$ the function (2.10) is the energy for \mathbf{Z}_2 electrodynamics [11]. By making use of the numbers $\sigma(s_i^1) = (-1)^{\sigma^1(s_i^1)} = \pm 1$ it is possible to rewrite the function (2.10), $p = 1$ in the form (2.11) with the products of four numbers $\sigma(s_i^1)$. The numbers \mathbf{Z}_2^{add} and the algebraic topology notations allow us to consider Ising model and \mathbf{Z}_2 electrodynamics together. The Ising model and the \mathbf{Z}_2 electrodynamics are the mathematical models of the ferromagnetic crystals. From the algebraic point of view these models are similar. The magnetism is connected with the currents flowing along the closed contours. The expression (2.10) gives the energy of the \mathbf{Z}_2 - currents flowing along the closed contours ∂s_i^{p+1} . For the \mathbf{Z}_2 electrodynamics ($p = 1$) the closed contour ∂s_i^2 consists in general of four boundary edges of the face s_i^2 . For the Ising model ($p = 0$) the closed contour ∂s_i^1 consists in general of two boundary vertices of the edge s_i^1 . It seems that the \mathbf{Z}_2 electrodynamics has more physical sense than the Ising model.

The equality (2.9) implies

$$\sum_{\sigma^p \in C^p(P(G(M)), \mathbf{Z}_2^{add})} \exp\{-\beta \overline{H}'_0(\partial^* \sigma^p)\} = Z_p(J, 0; G(M)) \exp\{-\beta D\},$$

$$Z_p(J, 0; G(M)) = \sum_{\sigma^p \in C^p(P(G(M)), \mathbf{Z}_2^{add})} \exp\{-\beta \bar{H}_0(\partial^* \sigma^p)\}. \quad (2.12)$$

The function (2.12) is the partition function of Ising model ($p = 0$) and of \mathbf{Z}_2 electrodynamics ($p = 1$) in the absence of magnetic field.

Let the cochain $\chi^p \in C^p(P(G(M)), \mathbf{Z}_2^{add})$ take the value $1 \in \mathbf{Z}_2^{add}$ at the cells s_1^p, \dots, s_m^p and be equal to $0 \in \mathbf{Z}_2^{add}$ at all other p -dimensional cells of the graph $G(M)$. The function

$$\begin{aligned} \alpha(\chi^p; G(M)) = & \\ (Z_p(J, 0; G(M)) \exp\{-\beta D\})^{-1} & \sum_{\sigma^p \in C^p(P(G(M)), \mathbf{Z}_2^{add})} (-1)^{\langle \chi^p, \sigma^p \rangle} \exp\{-\beta \bar{H}'_0(\partial^* \sigma^p)\} = \\ (Z_p(J, 0; G(M)))^{-1} & \sum_{\sigma^p \in C^p(P(G(M)), \mathbf{Z}_2^{add})} (-1)^{\langle \chi^p, \sigma^p \rangle} \exp\{-\beta \bar{H}_0(\partial^* \sigma^p)\}, \end{aligned} \quad (2.13)$$

$(-1)^{\langle \chi^p, \sigma^p \rangle} = (-1)^{\sigma^p(s_1^p)} \dots (-1)^{\sigma^p(s_m^p)}$, is the correlation function at the cells s_1^p, \dots, s_m^p of the lattice $G(M)$. The definitions (1.13) and (2.13) are consistent. If the cochain 0 takes the value $0 \in \mathbf{Z}_2^{add}$ at any p -dimensional cell of the lattice $G(M)$, then the correlation function $\alpha(0; G(M)) = 1$. The function (2.13) is the correlation function of Ising model ($p = 0$) and of \mathbf{Z}_2 electrodynamics ($p = 1$) in the absence of magnetic field. For the particular values of the interaction energies the correlation functions of the three-dimensional \mathbf{Z}_2 electrodynamics with free boundary conditions are calculated in the paper [11]. These correlation functions are connected with the correlation functions of the two-dimensional Ising model. Below we calculate the correlation functions of the Ising model and the \mathbf{Z}_2 electrodynamics for the case when the sign of the interaction energy $J(s_i^{p+1})$ is independent of the cell s_i^{p+1} and the interaction energy $J(s_i^{p+1})$ depends on the cell s_i^{p+1} . The interaction energy $J(s_i^{p+1})$ is supposed to be small in contrast with the paper [8].

By making use of the harmonic analysis on the group $C^p(P(G(M)), \mathbf{Z}_2^{add})$ and the first relation (2.6) it is possible to prove ([12], Proposition 3.1)

$$Z_p(J, 0; G(M)) = 2^{\#(G;p)} \left(\prod_{s_i^{p+1} \in P(G(M))} \cosh \beta J(s_i^{p+1}) \right) Z_{r,p}(J, 0; G(M)), \quad (2.14)$$

$$Z_{r,p}(J, 0; G(M)) = \sum_{\xi^{p+1} \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add}), \partial \xi^{p+1} = 0} \|\xi^{p+1}\|_{J, G(M)}, \quad (2.15)$$

$$\alpha(\chi^p; G(M)) = (Z_{r,p}(J, 0; G(M)))^{-1} \sum_{\xi^{p+1} \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add}), \partial \xi^{p+1} = \chi^p} \|\xi^{p+1}\|_{J, G(M)}, \quad (2.16)$$

$$\|\xi^{p+1}\|_{J, G(M)} = \prod_{s_i^{p+1} \in P(G(M))} \left(\tanh \beta J(s_i^{p+1}) \right)^{\tau((-1)^{\xi^{p+1}(s_i^{p+1})})}, \quad (2.17)$$

$$\tau((-1)^\epsilon) = \frac{1 - (-1)^\epsilon}{2} = \begin{cases} 1, & \epsilon = 1 \in \mathbf{Z}_2^{add}, \\ 0, & \epsilon = 0 \in \mathbf{Z}_2^{add}. \end{cases} \quad (2.18)$$

The constant $\#(G(M); p)$ is the total number of the p -dimensional cells of the cell complex $P(G(M))$. The correlation function (2.16) is equal to zero for $\chi^p \notin B_p(P(G(M)), \mathbf{Z}_2^{add})$.

For the cochain $\chi^0 \in B_0(P(G(M'_1; M_1)), \mathbf{Z}_2^{add})$ the equation $\partial\xi^1 = \chi^0$ has the unique solution. The group of cycles $Z_1(P(G(M'_1; M_1)), \mathbf{Z}_2^{add})$ consists of the cochain 0. The equalities (2.15), (2.16) imply

$$Z_{r,0}(J, 0; G(M'_1; M_1)) = 1, \quad (2.19)$$

$$\alpha(\chi^0; G(M'_1; M_1)) = \|\xi^1\|_{J, G(M'_1; M_1)}. \quad (2.20)$$

For the cochain $\chi^0 \notin B_0(P(G(M'_1; M_1)), \mathbf{Z}_2^{add})$ the equation $\partial\xi^1 = \chi^0$ has no solutions. The equality (2.16) implies

$$\alpha(\chi^0; G(M'_1; M_1)) = 0, \quad \chi^0 \notin B_0(P(G(M'_1; M_1)), \mathbf{Z}_2^{add}). \quad (2.21)$$

For the cochain $\chi^1 \in B_1(P(G(M'_1, M'_2; M_1, M_2)), \mathbf{Z}_2^{add})$. The equation $\partial\xi^2 = \chi^1$ has unique solution. The group of cycles $Z_2(P(G(M'_1, M'_2; M_1, M_2)), \mathbf{Z}_2^{add})$ consists of the cochain 0. The equalities (2.15), (2.16) imply

$$Z_{r,1}(J, 0; G(M'_1, M'_2; M_1, M_2)) = 1, \quad (2.22)$$

$$\alpha(\chi^1; G(M'_1, M'_2; M_1, M_2)) = \|\xi^2\|_{J, G(M'_1, M'_2; M_1, M_2)}. \quad (2.23)$$

For the cochain $\chi^1 \notin B_1(P(G(M'_1, M'_2; M_1, M_2)), \mathbf{Z}_2^{add})$ the equation $\partial\xi^2 = \chi^1$ has no solutions. The equality (2.16) implies

$$\alpha(\chi^1; G(M'_1, M'_2; M_1, M_2)) = 0, \quad \chi^1 \notin B_1(P(G(M'_1, M'_2; M_1, M_2)), \mathbf{Z}_2^{add}). \quad (2.24)$$

The partition function (2.14) for $p = 0, d = 2$ was "obtained" by Kac and Ward [13]:
 "The partition function of the two-dimensional square net Ising model can be easily put in the form [14]

$$(\cosh \beta J_2)^h (\cosh \beta J_1)^v \sum g(l, k) x^l y^k, \quad (2.25)$$

where

$$x = \tanh \beta J_2, \quad y = \tanh \beta J_1,$$

h the total number of horizontal links, v the total number of vertical links, and $g(l, k)$ the number of "closed polygons" with l horizontal and k vertical links."

Let us compare the expression (2.14) with the expression (2.25):

$$(\cosh \beta J_2)^h (\cosh \beta J_1)^v \sum g(l, k) (\tanh \beta J_2)^l (\tanh \beta J_1)^k = \left(\prod_{s_i^{p+1} \in P(G(M))} \cosh \beta J(s_i^{p+1}) \right) \sum_{\xi^1 \in C^1(P(G(M)), \mathbf{Z}_2^{add}), \partial\xi^1 = 0} \|\xi^1\|_{J, G(M)}. \quad (2.26)$$

The interaction energy $J(s_i^1)$ depends on the orientation of the edge s_i^1 only. The normalization constant $2^{\#(G(M);p)} = (\#\{\epsilon \in \mathbf{Z}_2^{add}\})^{\#(G(M);p)}$ for the harmonic analysis on the group $C^p(P(G(M)), \mathbf{Z}_2^{add})$ is missed in the expression (2.26).

There is no any expression for the partition function of Ising model in the paper [14]. Van der Waerden believed that the sum with the "long order" [14]

$$\sum g(l, k) z^{l+k}, \quad z = \exp\{-\beta J\} \neq \tanh \beta J \quad (2.27)$$

is important to study the crystals. $g(l, k)$ is the number of closed polygons with l horizontal and k vertical links. It seems that the definitions of the number $g(l, k)$ in the papers [13]

and [14] are different. Van der Waerden did not use the modulo 2 residuals. Kac, Ward [13] and van der Waerden [14] did not consider the correlation functions and avoided to use the algebraic topology notations. The relation (2.16) needs the algebraic topology notations.

Let the edge $\{(k_1, k_2), (l_1, l_2)\}$ have the end vertices $(k_1, k_2), (l_1, l_2) \in G(M'_1, M'_2; M_1, M_2)$. The oriented edge $((k_1, k_2), (l_1, l_2))$ has the initial vertex (k_1, k_2) and the final vertex (l_1, l_2) . Kac and Ward [13]: "In the main body of the paper we shall explain in detail the method of computing which yields the partition function up to negligible terms due to boundary effects. Several combinatorial points will be dealt with a heuristic manner only. We do not go into the details of rigor because our main aim is not so much an alternative derivation of the Onsager-Kaufman formula but a demonstration that a combinatorial approach is indeed possible." Kac and Ward [13] discussed the following formula for the partition function (2.15)

$$Z_{r,0}(J, 0; G(M'_1, M'_2; M_1, M_2)) = \det(I + T(J)) \quad (2.28)$$

where I is the identity matrix on the set of the oriented edges $((k_1, k_2), (l_1, l_2))$ and the interaction matrix

$$\begin{aligned} T(J)_{(((k_1, k_2), (l_1, l_2))), ((k'_1, k'_2), (l'_1, l'_2)))} &= 0, \quad (l_1, l_2) \neq (k'_1, k'_2), \\ T(J)_{(((k_1, k_2), (l_1, l_2))), ((l_1, l_2), (k'_1, k'_2)))} &= 0, \quad (k_1, k_2) \neq (k'_1, k'_2), \\ T(J)_{(((k_1, k_2), (l_1, l_2))), ((l_1, l_2), (k'_1, k'_2)))} &= \tanh(\beta J(\{(k_1, k_2), (l_1, l_2)\})) \times \\ \exp\left\{\frac{i}{2} [(l_1 - k_1, l_2 - k_2), (k'_1 - l_1, k'_2 - l_2)]\right\}, & \quad (k_1, k_2) \neq (k'_1, k'_2). \end{aligned} \quad (2.29)$$

For a vertical edge $J(\{(k_1, k_2), (k_1, k_2 + 1)\}) = J_1$ and for a horizontal edge $J(\{(k_1, k_2), (k_1 + 1, k_2)\}) = J_2$. The number $[(l_1, l_2), (l'_1, l'_2)]$ is the minimal radian measure of the angle between the direction of the vector (l_1, l_2) and the direction of the vector (l'_1, l'_2) . For an arbitrary finite connected graph G on the lattice $\mathbf{Z}^{\times 2}$ the following formula

$$Z_{r,0}^2(J, 0; G) = \det(I - T(J)) \quad (2.30)$$

is proved in the paper [15]. By making use of the formulae (2.16), (2.30) the correlation functions of the two-dimensional Ising model with the free boundary conditions are obtained in the paper [12]. The formula (2.30) implies the alternative derivation [12] of the Onsager-Kaufman formula. For the periodic boundary conditions McCoy and Wu [10] represented the partition function (2.15) as the linear combination of Pfaffians. The counterexample for the McCoy-Wu formula [10] was constructed in the paper [16]. The Euler characteristic of the orientable two-dimensional sphere S^2 is equal to 2. It implies the simple proof of the formula (2.30) in the paper [16]. The definition (2.29) uses the plane lattice crucially. The formula (2.30) for the three-dimensional Ising model is not clear.

The spontaneous magnetization (1.16) depends on the correlation functions only. By making use of the formula (2.16) we shall obtain the correlation functions of the d - dimensional Ising model ($d = 2, 3$) and of the three-dimensional \mathbf{Z}_2 electrodynamics with the free boundary conditions without calculation of the partition functions (2.15). In order to calculate the correlation functions (2.16) we need the notion of the connected cochain.

The set of the cells s_i^{p+1} on which the cochain $\zeta^{p+1} \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$ takes the value 1 $\in \mathbf{Z}_2^{add}$ is called the support of the cochain ζ^{p+1} . The nonzero cochain $\zeta^{p+1} \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$ is called connected if for any two cells s_i^{p+1}, s_j^{p+1} from the support of

the cochain ζ^{p+1} there exists a connecting sequence of the cells $s_i^{p+1} = s_1^{p+1}, s_2^{p+1}, \dots, s_{l-1}^{p+1}, s_l^{p+1} = s_j^{p+1}$ from the support of the cochain ζ^{p+1} with the common boundary cells:

$$s_k^p : (s_k^{p+1} : s_k^p)(s_{k+1}^{p+1} : s_k^p) = 1 \in \mathbf{Z}_2^{add}, \quad k = 1, \dots, l-1. \quad (2.31)$$

Any nonzero cochain $\zeta^{p+1} \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$ can be uniquely represented as the sum of the connected nonzero cochains $\zeta_m^{p+1} \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$, $m = 1, \dots, k$. For $m \neq n$ the cells from the supports of the cochains $\zeta_m^{p+1}, \zeta_n^{p+1}$ have no the common boundary cells:

$$(s_i^{p+1} : s_k^p)(s_j^{p+1} : s_k^p) = 0 \in \mathbf{Z}_2^{add}, \quad \zeta_m^{p+1}(s_i^{p+1}) = 1, \quad \zeta_n^{p+1}(s_j^{p+1}) = 1, \quad m \neq n, \quad (2.32)$$

$m, n = 1, \dots, k$. For the nonzero cochain $\chi^p \in B_p(P(G(M)), \mathbf{Z}_2^{add})$ any solution of the equation $\partial \xi^{p+1}(M) = \chi^p$ can be uniquely represented as

$$\xi^{p+1} = \sum_{i_1=1}^{k_1} \lambda_{i_1}^{p+1}(M) + \xi_1^{p+1}. \quad (2.33)$$

The connected cochains $\lambda_{i_1}^{p+1}(M) \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$, $i_1 = 1, \dots, k_1$, satisfy the equations

$$\partial \left(\sum_{i_1=1}^{k_1} \lambda_{i_1}^{p+1}(M) \right) = \chi^p; \quad \partial \lambda_{i_1}^{p+1}(M) \neq 0;$$

$$(s_i^{p+1} : s_l^p)(s_j^{p+1} : s_l^p) = 0, \quad \lambda_{i_1}^{p+1}(M)(s_i^{p+1}) = 1, \quad \lambda_{j_1}^{p+1}(M)(s_j^{p+1}) = 1, \quad i_1 \neq j_1, \quad (2.34)$$

$i_1, j_1 = 1, \dots, k_1$. The cochain $\xi_1^{p+1} \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$ satisfies the equations

$$\partial \xi_1^{p+1} = 0;$$

$$(s_i^{p+1} : s_l^p)(s_j^{p+1} : s_l^p) = 0, \quad \xi_1^{p+1}(s_i^{p+1}) = 1, \quad \lambda_{i_1}^{p+1}(M)(s_j^{p+1}) = 1, \quad i_1 = 1, \dots, k_1. \quad (2.35)$$

The cochain χ^p does not determine the integer k_1 in the relations (2.33) - (2.35). The integer k_1 does not exceed the total number of cells in the support of the cochain χ^p . The relations (2.17), (2.34), (2.35) imply

$$\left\| \sum_{i_1=1}^{k_1} \lambda_{i_1}^{p+1}(M) + \xi_1^{p+1} \right\|_{J,G(M)} = \left(\prod_{i_1=1}^{k_1} \|\lambda_{i_1}^{p+1}(M)\|_{J,G(M)} \right) \|\xi_1^{p+1}\|_{J,G(M)}. \quad (2.36)$$

In view of the equality (2.33) the relation (2.16) may be rewritten for the nonzero cochain $\chi^p \in B_{p+1}(P(G(M)), \mathbf{Z}_2^{add})$ as

$$\begin{aligned} \alpha(\chi^p; G(M)) &= \sum_{1 \leq k_1} \sum_{\lambda_{i_1}^{p+1}(M): (2.34)} \left(\prod_{i_1=1}^{k_1} \|\lambda_{i_1}^{p+1}(M)\|_{J,G(M)} \right) \\ &\times \left(1 + \alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{i_1}^{p+1}(M); G(M) \right) \right)^{-1}, \\ &\left(1 + \alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{i_1}^{p+1}(M); G(M) \right) \right)^{-1} = \end{aligned} \quad (2.37)$$

$$(Z_{r,p}(J, 0; G(M)))^{-1} \sum_{\xi_1^{p+1}: (2.35)} \|\xi_1^{p+1}\|_{J, G(M)}. \quad (2.38)$$

The cochain $\xi^{p+1} \in Z_{p+1}(P(G(M)), \mathbf{Z}_2^{add})$ may have the form similar to the sum (2.33)

$$\xi^{p+1} = \sum_{i_2=1}^{k_2} \lambda_{2i_2}^{p+1}(M) + \xi_2^{p+1}. \quad (2.39)$$

The connected cochains $\lambda_{2i_2}^{p+1}(M) \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$, $i_2 = 1, \dots, k_2$, satisfy the equations

$$\partial \lambda_{2i_2}^{p+1}(M) = 0;$$

$$\begin{aligned} \forall i_2 \exists i_1, i, j, l: (s_i^{p+1} : s_l^p)(s_j^{p+1} : s_l^p) = 1, \lambda_{2i_2}^{p+1}(M)(s_i^{p+1}) = 1, \lambda_{1i_1}^{p+1}(M)(s_j^{p+1}) = 1; \\ (s_i^{p+1} : s_l^p)(s_j^{p+1} : s_l^p) = 0, \lambda_{2i_2}^{p+1}(M)(s_i^{p+1}) = 1, \lambda_{2j_2}^{p+1}(M)(s_j^{p+1}) = 1, i_2 \neq j_2, \end{aligned} \quad (2.40)$$

$i_2, j_2 = 1, \dots, k_2$. The cochain $\xi_2^{p+1} \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$ satisfies the equations

$$\partial \xi_2^{p+1} = 0; (s_i^{p+1} : s_l^p)(s_j^{p+1} : s_l^p) = 0,$$

$$\xi_2^{p+1}(s_i^{p+1}) = 1, \lambda_{ni_n}^{p+1}(M)(s_j^{p+1}) = 1, i_n = 1, \dots, k_n, n = 1, 2. \quad (2.41)$$

If the cochains $\lambda_{2i_2}^{p+1}(M) = 0$, then the equations (2.41) coincide with the equations (2.35) and the cochain (2.39) coincides with the cochain ξ_1^{p+1} satisfying the equations (2.35). The relations (2.40), (2.41) imply for the cochain (2.39) the relation similar to the relation (2.36). The relations (2.15), (2.35), (2.38) - (2.41) imply

$$Z_{r,p}(J, 0; G(M)) = \sum_{\xi_1^{p+1}: (2.35)} \|\xi_1^{p+1}\|_{J, G(M)} +$$

$$\sum_{1 \leq k_2} \sum_{\lambda_{2i_2}^{p+1}(M): (2.40)} \left(\prod_{i_2=1}^{k_2} \|\lambda_{2i_2}^{p+1}(M)\|_{J, G(M)} \right) \sum_{\xi_2^{p+1}: (2.41)} \|\xi_2^{p+1}\|_{J, G(M)}, \quad (2.42)$$

$$\begin{aligned} \alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(M); G(M) \right) = \sum_{1 \leq k_2} \sum_{\lambda_{2i_2}^{p+1}(M): (2.40)} \left(\prod_{i_2=1}^{k_2} \|\lambda_{2i_2}^{p+1}(M)\|_{J, G(M)} \right) \\ \times \left(1 + \alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(M), \sum_{i_2=1}^{k_2} \lambda_{2i_2}^{p+1}(M); G(M) \right) \right)^{-1}, \end{aligned} \quad (2.43)$$

$$\begin{aligned} \left(1 + \alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(M), \sum_{i_2=1}^{k_2} \lambda_{2i_2}^{p+1}(M); G(M) \right) \right)^{-1} = \\ \left(\sum_{\xi_1^{p+1}: (2.35)} \|\xi_1^{p+1}\|_{J, G(M)} \right)^{-1} \sum_{\xi_2^{p+1}: (2.41)} \|\xi_2^{p+1}\|_{J, G(M)}. \end{aligned} \quad (2.44)$$

We continue this process to construct the sequence of the connected cochains $\lambda_{ni_n}^{p+1}(M)$, $i_n = 1, \dots, k_n$, $n = 1, 2, \dots$, from the group $C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$ satisfying the equations (2.34) for $n = 1$, (2.40) for $n = 2$ and the equations

$$(s_i^{p+1} : s_l^p)(s_j^{p+1} : s_l^p) = 0, \lambda_{ni_n}^{p+1}(M)(s_i^{p+1}) = 1, \lambda_{mi_m}^{p+1}(M)(s_j^{p+1}) = 1,$$

$$\begin{aligned}
i_n &= 1, \dots, k_n, \quad i_m = 1, \dots, k_m, \quad m = 1, \dots, n-2 \geq 1; \\
\forall i_n \exists i_{n-1}, i, j, l : & (s_i^{p+1} : s_l^p)(s_j^{p+1} : s_l^p) = 1, \\
\lambda_{ni_n}^{p+1}(M)(s_i^{p+1}) &= 1, \quad \lambda_{(n-1)i_{n-1}}^{p+1}(M)(s_j^{p+1}) = 1, \quad n \geq 2; \\
\partial \lambda_{ni_n}^{p+1}(M) &= 0; \quad (s_i^{p+1} : s_l^p)(s_j^{p+1} : s_l^p) = 0, \\
\lambda_{ni_n}^{p+1}(M)(s_i^{p+1}) &= 1, \quad \lambda_{nj_n}^{p+1}(M)(s_j^{p+1}) = 1, \quad i_n \neq j_n, \quad i_n, j_n = 1, \dots, k_n, \quad n \geq 2.
\end{aligned} \tag{2.45}$$

We construct also the sequence of the cochains ξ_n^{p+1} , $n = 1, 2, \dots$, satisfying the equations (2.35) for $n = 1$, (2.41) for $n = 2$ and the equations

$$\begin{aligned}
\partial \xi_n^{p+1} &= 0; \quad (s_i^{p+1} : s_l^p)(s_j^{p+1} : s_l^p) = 0, \\
\xi_n^{p+1}(M)(s_i^{p+1}) &= 1, \quad \lambda_{mi_m}^{p+1}(M)(s_j^{p+1}) = 1, \quad i_m = 1, \dots, k_m, \quad m = 1, \dots, n \geq 1.
\end{aligned} \tag{2.46}$$

Similarly to the relations (2.43), (2.44) we have the anti-recurrent relations for $n \geq 2$

$$\begin{aligned}
& \alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(M), \dots, \sum_{i_n=1}^{k_n} \lambda_{ni_n}^{p+1}(M); G(M) \right) = \\
& \sum_{1 \leq k_{n+1}} \sum_{\substack{\lambda_{(n+1)i_{n+1}}^{p+1}(M): \\ (2.45), \quad n \rightarrow n+1}} \left(\prod_{i_{n+1}=1}^{k_{n+1}} \|\lambda_{(n+1)i_{n+1}}^{p+1}(M)\|_{P(G(M))} \right) \\
& \times \left(1 + \alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(M), \dots, \sum_{i_{n+1}=1}^{k_{n+1}} \lambda_{(n+1)i_{n+1}}^{p+1}(M); G(M) \right) \right)^{-1},
\end{aligned} \tag{2.47}$$

$$\begin{aligned}
& \left(1 + \alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(M), \dots, \sum_{i_{n+1}=1}^{k_{n+1}} \lambda_{(n+1)i_{n+1}}^{p+1}(M); G(M) \right) \right)^{-1} = \\
& \left(\sum_{\xi_n^{p+1}: (2.46)} \|\xi_n^{p+1}\|_{J, G(M)} \right)^{-1} \sum_{\xi_{n+1}^{p+1}: (2.46), \quad n \rightarrow n+1} \|\xi_{n+1}^{p+1}\|_{J, G(M)}.
\end{aligned} \tag{2.48}$$

For the finite graph $G(M)$ the group $C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$ contains the finite number of the cochains. Let $N(\lambda, G(M)) + 1$ be the maximal number of the cochain in the cochain sequence satisfying the equations (2.34), (2.40), (2.45). Let $\lambda_{N(\lambda, G(M)) + 2}^{p+1}(M)$ be the connected cochain satisfying the first and the third equations (2.45). If the second equality (2.45) for $n = N(\lambda, G(M)) + 2$ holds, then we have constructed the sequence of the cochains $\lambda_{1i_1}^{p+1}(M), \dots, \lambda_{(n+1)i_{n+1}}^{p+1}(M)$, $\lambda_{n+2}^{p+1}(M)$, $n = N(\lambda, G(M))$, satisfying the equations (2.34), (2.40), (2.45). This sequence of the cochains does not exist. Therefore the second equality (2.45) for $n = N(\lambda, G(M)) + 2$ is not valid and the group of the cochains (2.46) for $n = N(\lambda, G(M))$ coincides with the group of the cochains (2.46) for $n = N(\lambda, G(M)) + 1$. Now the relation (2.48) implies

$$\alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(M), \dots, \sum_{i_{n+1}=1}^{k_{n+1}} \lambda_{(n+1)i_{n+1}}^{p+1}(M); G(M) \right) = 0, \quad n = N(\lambda, G(M)). \tag{2.49}$$

The anti-recurrent relations (2.37), (2.43), (2.47), (2.49) define the correlation functions for the finite graph $G(M)$.

The length of $\xi^{p+1} \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$ is the number of the cells in the support

$$|\xi^{p+1}|_{G(M)} = \sum_{s_i^{p+1} \in P(G(M))} \tau((-1)^{\xi^{p+1}(s_i^{p+1})}). \quad (2.50)$$

$\tau((-1)^\epsilon)$ is given by the definition (2.18). The homology group triviality implies the coincidence of the groups $Z_{p+1}(P(G(M)), \mathbf{Z}_2^{add})$ and $B_{p+1}(P(G(M)), \mathbf{Z}_2^{add})$ for the graph $G(M) = G(M'_1, \dots, M'_d; M_1, \dots, M_d)$. Let us compute the parity of the number (2.50) for the cochain $\partial\xi^{p+2} \in B_{p+1}(P(G(M)), \mathbf{Z}_2^{add})$. Any cell s_i^{p+2} from the support of the cochain ξ^{p+2} has $2(p+2)$ boundary cells s_j^{p+1} . Let the cell s_j^{p+1} belong to the boundaries of $2m+1$ cells s_i^{p+2} from the support of the cochain ξ^{p+2} . In order to get the number $2(p+2)|\xi^{p+2}|_{P(G(M))}$ we count the cell s_j^{p+1} exactly $2m+1$ times. The cell s_j^{p+1} should be included into the support of the cochain $\partial\xi^{p+2}$. Let the cell s_j^{p+1} belong to the boundaries of $2m$ cells s_i^{p+2} from the support of the cochain ξ^{p+2} . In order to get the number $2(p+2)|\xi^{p+2}|_{P(G(M))}$ we count the cell s_j^{p+1} exactly $2m$ times. The cell s_j^{p+1} should be excluded from the support of the cochain $\partial\xi^{p+2}$. The parities of the numbers $|\partial\xi^{p+2}|_{G(M)}$ and $2(p+2)|\xi^{p+2}|_{P(G(M))}$ coincide

$$|\partial\xi^{p+2}|_{G(M)} = 0 \pmod{2}. \quad (2.51)$$

Let the sign of the interaction energy $J(s_i^{p+1})$ be independent of the cell s_i^{p+1} . The equalities (2.17), (2.51) for a cochain $\xi^{p+1} \in Z_{p+1}(P(G(M)), \mathbf{Z}_2^{add}) = B_{p+1}(P(G(M)), \mathbf{Z}_2^{add})$ imply

$$||\xi^{p+1}||_{J,G(M)} = \prod_{s_i^{p+1} \in P(G(M))} \left(\tanh \beta J(s_i^{p+1}) \right)^{\tau((-1)^{\xi^{p+1}(s_i^{p+1})})} \geq 0. \quad (2.52)$$

$\tau((-1)^\epsilon)$ is given by the definition (2.18). The definitions (2.43), (2.47), (2.49) and the inequality (2.52) imply

$$\alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(M), \dots, \sum_{i_n=1}^{k_n} \lambda_{ni_n}^{p+1}(M); G(M) \right) \geq 0, \quad n = 1, \dots, N(\lambda, G(M)) + 1. \quad (2.53)$$

Let us estimate the number of the connected cochains λ^{p+1} with the value $\lambda^{p+1}(s_j^{p+1}) = 1 \in \mathbf{Z}_2^{add}$ at the fixed cell s_j^{p+1} . Let s_l^p be a boundary cell of the cell s_j^{p+1} . In order to construct a new $(p+1)$ -dimensional cell of the graph $G(M'_1, \dots, M'_d; M_1, \dots, M_d)$ with the boundary cell s_l^p we need to choose a vertex of the cell s_l^p and one of $2(d-p)-1$ edges orthogonal to the cell s_l^p . One edge orthogonal to the cell s_l^p corresponds with the fixed cell s_j^{p+1} . Any number $1, \dots, 2(d-p)-1$ of the new $(p+1)$ -dimensional cells with the boundary cell s_l^p may belong to the support of the cochain λ^{p+1} . Due to the Newton binomial formula the possible number of these sets of the cells from the support of the cochain λ^{p+1} is equal to

$$\sum_{k=0}^{2(d-p)-1} \frac{(2(d-p)-1)!}{k!(2(d-p)-1-k)!} - 1 = 2^{2(d-p)-1} - 1. \quad (2.54)$$

The number (2.54) implies the estimation

$$\#\{\text{connected } \lambda^{p+1} : \lambda^{p+1}(s_j^{p+1}) = 1\} < 2^{(2(d-p)-1)(|\lambda^{p+1}|_{G(M)}-1)}. \quad (2.55)$$

If the sign of the interaction energy $J(s_i^{p+1})$ is independent of the cell s_i^{p+1} and the interaction energy $J(s_i^{p+1})$ satisfies the inequality

$$|\tanh \beta J(s_i^{p+1})| < 2^{2(p-d)+1}, \quad (2.56)$$

then the inequalities (2.53), (2.55) imply that the sums (2.37), (2.43) and (2.47) are bounded by the constants independent of the graph $G(M)$.

Let us prove that the sequence of the correlation functions $\alpha(\chi^p; G(M))$ is the convergent Cauchy sequence when $G(M) \rightarrow \mathbf{Z}^{\times d}$. Let the graph $G(N'_1, \dots, N'_d; N_1, \dots, N_d)$ be the subset of the graph $G(M'_1, \dots, M'_d; M_1, \dots, M_d)$: $M'_i < N'_i$, $N_i < M_i$, $i = 1, \dots, d$. The connected cochain $\lambda_{n_{in}}^{p+1}(M) \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$ coincides with the connected cochains $\lambda_{n_{in}}^{p+1}(N) \in C^{p+1}(P(G(N)), \mathbf{Z}_2^{add})$ when all the cells from its supports belong to the graph $G(N)$. The equation (2.37) for the nonzero cochain $\chi^p \in B_{p+1}(P(G(N)), \mathbf{Z}_2^{add})$ implies

$$\alpha(\chi^p; G(M)) - \alpha(\chi^p; G(N)) = \sum_{1 \leq k_1} \sum_{\lambda_{1i_1}^{p+1}(N) \in C^{p+1}(P(G(N)), \mathbf{Z}_2^{add})} \quad (2.34)$$

$$\begin{aligned} & \left(\prod_{i_1=1}^{k_1} \|\lambda_{1i_1}^{p+1}(N)\|_{J, G(N)} \right) \left(1 + \alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(N); G(M) \right) \right)^{-1} \\ & \times \left(1 + \alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(N); G(N) \right) \right)^{-1} \left(\alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(N); G(N) \right) \right. \\ & \left. - \alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(N); G(M) \right) \right) + \sum_{1 \leq k_1} \sum_{\substack{\lambda_{1i_1}^{p+1}(M) \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add}), \\ \lambda_{1i_1}^{p+1}(M) \notin C^{p+1}(P(G(N)), \mathbf{Z}_2^{add})}} \quad (2.34) \\ & \left(\prod_{i_1=1}^{k_1} \|\lambda_{1i_1}^{p+1}(M)\|_{J, G(M)} \right) \left(1 + \alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(M); G(M) \right) \right)^{-1}. \end{aligned} \quad (2.57)$$

The equations (2.43), (2.47) imply

$$\begin{aligned} & \alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(N), \dots, \sum_{i_n=1}^{k_n} \lambda_{ni_n}^{p+1}(N); G(M) \right) \\ & - \alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(N), \dots, \sum_{i_n=1}^{k_n} \lambda_{ni_n}^{p+1}(N); G(N) \right) = \\ & \sum_{1 \leq k_{n+1}} \sum_{\lambda_{(n+1)i_{n+1}}^{p+1}(N): (2.45), n \rightarrow n+1} \left(\prod_{i_{n+1}=1}^{k_{n+1}} \|\lambda_{(n+1)i_{n+1}}^{p+1}(N)\|_{P(G(N))} \right) \\ & \times \left(1 + \alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(N), \dots, \sum_{i_{n+1}=1}^{k_{n+1}} \lambda_{(n+1)i_{n+1}}^{p+1}(N); G(M) \right) \right)^{-1} \\ & \times \left(1 + \alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(N), \dots, \sum_{i_{n+1}=1}^{k_{n+1}} \lambda_{(n+1)i_{n+1}}^{p+1}(N); G(N) \right) \right)^{-1} \end{aligned}$$

$$\begin{aligned}
& \times \left(\alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(N), \dots, \sum_{i_{n+1}=1}^{k_{n+1}} \lambda_{(n+1)i_{n+1}}^{p+1}(N); G(N) \right) \right. \\
& \left. - \alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(N), \dots, \sum_{i_{n+1}=1}^{k_{n+1}} \lambda_{(n+1)i_{n+1}}^{p+1}(N); G(M) \right) \right) \\
& + \sum_{1 \leq k_{n+1}} \sum_{\substack{\lambda_{n+1}^{p+1}(M) \notin C^{p+1}(P(G(N)), \mathbf{Z}_2^{add}): \\ (2.45), n \rightarrow n+1}} \left(\prod_{i_{n+1}=1}^{k_{n+1}} \|\lambda_{(n+1)i_{n+1}}^{p+1}(M)\|_{P(G(M))} \right) \\
& \times \left(1 + \alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(N), \dots, \sum_{i_n=1}^{k_n} \lambda_{ni_n}^{p+1}(N), \sum_{i_{n+1}=1}^{k_{n+1}} \lambda_{(n+1)i_{n+1}}^{p+1}(M); G(M) \right) \right)^{-1} \quad (2.58)
\end{aligned}$$

for $n = 1, \dots, N(\lambda, G(M))|_{M=N}$. The relations (2.47), (2.49) imply

$$\begin{aligned}
& \alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(N), \dots, \sum_{i_{n+1}=1}^{k_{n+1}} \lambda_{(n+1)i_{n+1}}^{p+1}(N); G(M) \right) \\
& - \alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(N), \dots, \sum_{i_{n+1}=1}^{k_{n+1}} \lambda_{(n+1)i_{n+1}}^{p+1}(N); G(N) \right) = \\
& \sum_{1 \leq k_{n+2}} \sum_{\substack{\lambda_{(n+2)i_{n+2}}^{p+1}(M) \notin C^{p+1}(P(G(N)), \mathbf{Z}_2^{add}): \\ (2.45), n \rightarrow n+2}} \left(\prod_{i_{n+2}=1}^{k_{n+2}} \|\lambda_{(n+2)i_{n+2}}^{p+1}(M)\|_{P(G(M))} \right) \\
& \times \left(1 + \alpha \left(\chi^p, \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(N), \dots, \sum_{i_{n+1}=1}^{k_{n+1}} \lambda_{(n+1)i_{n+1}}^{p+1}(N), \sum_{i_{n+2}=1}^{k_{n+2}} \lambda_{(n+2)i_{n+2}}^{p+1}(M); G(M) \right) \right)^{-1} \quad (2.59)
\end{aligned}$$

for $n = N(\lambda, G(M))|_{M=N}$. The inequality (2.53) and the equalities (2.57) - (2.59) imply

$$\begin{aligned}
& |\alpha(\chi^p; G(M)) - \alpha(\chi^p; G(N))| \leq \sum_{\lambda_1^{p+1}(M) \notin C^{p+1}(P(G(N)), \mathbf{Z}_2^{add}): (2.34)} \left(\prod_{i_1=1}^{k_1} \|\lambda_{1i_1}^{p+1}(M)\|_{J, G(M)} \right) + \\
& \sum_{\substack{\lambda_{1i_1}^{p+1}(N) \in C^{p+1}(P(G(N)), \mathbf{Z}_2^{add}), \\ \lambda_{2i_2}^{p+1}(M) \notin C^{p+1}(P(G(N)), \mathbf{Z}_2^{add}): (2.34), (2.40)}} \left(\prod_{i_1=1}^{k_1} \|\lambda_{1i_1}^{p+1}(N)\|_{J, G(N)} \right) \left(\prod_{i_2=1}^{k_2} \|\lambda_{2i_2}^{p+1}(M)\|_{J, G(M)} \right) + \dots \\
& + \sum_{\substack{\lambda_{1i_1}^{p+1}(N), \dots, \lambda_{(n+1)i_{n+1}}^{p+1}(N) \in C^{p+1}(P(G(N)), \mathbf{Z}_2^{add}), \\ (\lambda_{(n+2)i_{n+2}}^{p+1}(M) \notin C^{p+1}(P(G(N)), \mathbf{Z}_2^{add}): (2.34), (2.45))}} \left(\prod_{l=1}^{n+1} \prod_{i_l=1}^{k_l} \|\lambda_{li_l}^{p+1}(N)\|_{J, G(N)} \right) \\
& \times \left(\prod_{i_{n+2}=1}^{k_{n+2}} \|\lambda_{(n+2)i_{n+2}}^{p+1}(M)\|_{J, G(M)} \right), \quad n = N(\lambda, G(M))|_{M=N}. \quad (2.60)
\end{aligned}$$

For the last multiplier in the n - term ($n = 1, \dots, N(\lambda, G(M))|_{M=N} + 2$) of the right-hand side of the inequality (2.60) the cochain $\lambda_{ni_n}^{p+1}(M) \notin C^{p+1}(P(G(N)), \mathbf{Z}_2^{add})$. The sum

$$\sum_{l=1}^{n-1} |\lambda_{li_l}^{p+1}(N)|_{G(N)} + |\lambda_{ni_n}^{p+1}(M)|_{G(M)}$$

of the cochain lengths in the n - term of the right-hand side of the inequality (2.60) exceeds the minimal distance from the support of the cochain χ^p to the boundary of the graph $G(N)$. Now the inequalities (2.55), (2.56) and (2.60) imply that the left-hand side of the inequality (2.60) is small for the large graphs $G(M)$, $G(N)$: the sequence of the correlation functions $\alpha(\chi^p; G(M))$ is the convergent Cauchy sequence when $G(M) \rightarrow \mathbf{Z}^{\times d}$.

3 Magnetization

Let us consider the one-dimensional Ising model with the free boundary conditions. We rewrite the energy function (2.10) in the form (2.11). For $2N+1$ vertices s_k^0 , $k = -N, \dots, N$, we define the numbers $\sigma_k = (-1)^{\sigma^0(s_k^0)} = \pm 1$ usual for Ising model. The partition function of the Ising model with the constant $J(s_k^1) = J$, $k = -N-1, \dots, N$, and $H(s_k^0) = H$

$$Z_0(J, H; G(-N, N)) = \sum_{\substack{\sigma_k = \pm 1, k = -N, \dots, N, \\ \sigma_{-N-1} = \sigma_{N+1} = 1}} \exp\left\{\beta J \sum_{k=-N-1}^N \sigma_k \sigma_{k+1} + \beta H \sum_{k=-N}^N \sigma_k\right\}. \quad (3.1)$$

It is possible to rewrite the definition (3.1)

$$Z_0(J, H; G(-N, N)) = \text{Tr} \left(A^{2N} B \right) \quad (3.2)$$

by making use of the 2×2 - matrices

$$\begin{aligned} A_{1+\tau(\sigma_1), 1+\tau(\sigma_2)} &= \exp\left\{\beta J \sigma_1 \sigma_2 + \frac{\beta H}{2} (\sigma_1 + \sigma_2)\right\}, \\ B_{1+\tau(\sigma_1), 1+\tau(\sigma_2)} &= \exp\left\{\beta \left(J + \frac{H}{2}\right) (\sigma_1 + \sigma_2)\right\}, \end{aligned} \quad (3.3)$$

$\sigma_1, \sigma_2 = \pm 1$, the numbers $\tau(\pm 1)$ are given by the definition (2.18). The 2×2 - matrix A is

$$A = K \begin{pmatrix} \lambda_+(J, H) & 0 \\ 0 & \lambda_-(J, H) \end{pmatrix} K^{-1}, \quad (3.4)$$

$$K = \begin{pmatrix} e^{\beta J} (\lambda_+(J, H) - \exp\{\beta J - \beta H\}) & 1 \\ 1 & e^{\beta J} (\lambda_-(J, H) - \exp\{\beta J + \beta H\}) \end{pmatrix}, \quad (3.5)$$

$$\begin{aligned} K^{-1} &= (e^{4\beta J} (\sinh \beta H + (\sinh^2 \beta H + e^{-4\beta J})^{1/2})^2 + 1)^{-1} \times \\ &\begin{pmatrix} -e^{\beta J} (\lambda_-(J, H) - \exp\{\beta J + \beta H\}) & 1 \\ 1 & -e^{\beta J} (\lambda_+(J, H) - \exp\{\beta J - \beta H\}) \end{pmatrix}. \end{aligned} \quad (3.6)$$

The eigenvalues $\lambda_{\pm}(J, H)$ are given by the relations (1.6). The equalities (3.4) - (3.6) yield the partition function (3.2)

$$\begin{aligned} Z_0(J, H; G(-N, N)) &= (-e^{4\beta J} (\sinh \beta H + (\sinh^2 \beta H + e^{-4\beta J})^{1/2})^2 - 1)^{-1} \left\{ \lambda_+^{2N}(J, H) \right. \\ &\quad \times (\exp\{4\beta J + \beta H\} (\lambda_+(J, H) - \exp\{\beta J - \beta H\}) (\lambda_-(J, H) - \exp\{\beta J + \beta H\}) \\ &\quad + e^{\beta J} ((\lambda_-(J, H) - \exp\{\beta J + \beta H\}) - (\lambda_+(J, H) - \exp\{\beta J - \beta H\})) \\ &\quad - \exp\{-2\beta J - \beta H\}) + \lambda_-^{2N}(J, H) (e^{-\beta H} (\lambda_+(J, H) - \exp\{\beta J - \beta H\}) \\ &\quad \times (\lambda_-(J, H) - \exp\{\beta J + \beta H\}) + e^{\beta J} (\lambda_+(J, H) - \exp\{\beta J - \beta H\}) \\ &\quad \left. - e^{\beta J} (\lambda_-(J, H) - \exp\{\beta J + \beta H\}) - \exp\{2\beta J + \beta H\}) \right\}. \end{aligned} \quad (3.7)$$

For the periodic boundary conditions the matrix $B = A$ in the relation (3.2) and the partition function expression (1.6) is simple. The eigenvalues (1.6) satisfy the inequality

$$\left| \frac{\lambda_-(J, H)}{\lambda_+(J, H)} \right| = \frac{|1 - e^{-4\beta J}|}{(\cosh \beta H + (\sinh^2 \beta H + e^{-4\beta J})^{1/2})^2} = \frac{|1 - e^{4\beta J}|}{(e^{2\beta J} \cosh \beta H + (e^{4\beta J} \sinh^2 \beta H + 1)^{1/2})^2} < 1. \quad (3.8)$$

By making use of the equality (3.7) and the inequality (3.8) we have the same magnetization

$$\lim_{N \rightarrow \infty} (\beta(2N+1))^{-1} \frac{\partial}{\partial H} (\ln Z_0(J, H; G(-N, N))) = (\sinh^2 \beta H + \exp\{-4\beta J\})^{-1/2} \sinh \beta H \quad (3.9)$$

as the magnetization (1.8). For the vacuum ($J = 0$) the partition function (3.1) is

$$Z_0(0, H; G(-N, N)) = (2 \cosh \beta H)^{2N+1}. \quad (3.10)$$

Due to the relations (1.6), (3.7), (3.10) we obtain the same spontaneous magnetization

$$\lim_{N \rightarrow \infty} \frac{\partial}{\partial x} \left(\ln \frac{Z_0(J, (2N+2)^{-1/2} \beta^{-1} x; G(-N, N))}{Z_0(0, (2N+2)^{-1/2} \beta^{-1} x; G(-N, N))} \right)_{x = \tanh \beta H} = (\exp\{2\beta J\} - 1) \tanh \beta H \quad (3.11)$$

as the spontaneous magnetization (1.12). $2N+2$ is the total number of the edges of the cell complex $P(G(-N, N))$. Due to (2.13) the two-spin correlation function is

$$\begin{aligned} < \sigma_m \sigma_n >_{2N+1} = (Z_0(J, 0; G(-N, N)))^{-1} \\ \times \left(\sum_{\substack{\sigma_k = \pm 1, \quad k = -N, \dots, N, \\ \sigma_{-N-1} = \sigma_{N+1} = 1}} \sigma_m \sigma_n \exp\{\beta J \sum_{k=-N-1}^N \sigma_k \sigma_{k+1}\} \right), \quad m, n = -N, \dots, N. \end{aligned} \quad (3.12)$$

In view of the relations (3.1), (3.10), (3.12)

$$\sum_{m, n = -N, \dots, N, m \neq n} < \sigma_m \sigma_n >_{2N+1} = \beta^{-2} \frac{\partial^2}{\partial H^2} \left(\ln \frac{Z_0(J, H; G(-N, N))}{Z_0(0, H; G(-N, N))} \right)_{H=0}. \quad (3.13)$$

The relations (1.6), (3.7), (3.10), (3.13) imply

$$\lim_{N \rightarrow \infty} (2N+2)^{-1} \sum_{m, n = -N, \dots, N, m \neq n} < \sigma_m \sigma_n >_{2N+1} = \exp\{2\beta J\} - 1. \quad (3.14)$$

We choose the number $2N+2 = \#(G(-N, N); 1)$ in the left-hand side of the equality (3.14). It is possible to choose any number $2N+M$ for an independent of N number M . In view of the relation (3.14) the spontaneous magnetizations (1.16) and (3.11) are similar

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{\partial}{\partial x} \left(\ln \frac{Z_0(J, (2N+2)^{-1/2} \beta^{-1} x; G(-N, N))}{Z_0(0, (2N+2)^{-1/2} \beta^{-1} x; G(-N, N))} \right)_{x = \tanh \beta H} = \\ & (2 \tanh \beta H) \lim_{N \rightarrow \infty} (2N+2)^{-1} \sum_{m, n = -N, \dots, N, m < n} < \sigma_m \sigma_n >_{2N+1}. \end{aligned} \quad (3.15)$$

Below we prove the equalities similar to the equality (3.15) for the d - dimensional Ising model ($d = 1, 2, 3$) with the free boundary conditions and with the interaction energy $J(s_i^1)$ depending on the edge s_i^1 . We obtain also the equalities similar to the equality (3.15) for the d - dimensional \mathbf{Z}_2 electrodynamics ($d = 2, 3$) with the free boundary conditions and with the interaction energy $J(s_i^2)$ depending on the face s_i^2 .

The partition function with the constant magnetic field $H(s_i^p) = H$

$$Z_p(J, H; G(M)) = \sum_{\sigma^p \in C^p(P(G(M)), \mathbf{Z}_2^{add})} \exp\{-\beta \bar{H}_0(\partial^* \sigma^p) + \beta H \sum_{s_i^p \in P(G(M))} (-1)^{\sigma^p(s_i^p)}\} \quad (3.16)$$

is similar to the partition function (3.1). For $p = 1$ it is possible to consider the magnetic field $H(s_i^1)$ depending on the edge s_i^1 orientation. For $H = 0$ the partition function (3.16) coincides with the partition function (2.12). By making use of the decomposition (2.8)

$$\exp\{\beta H(-1)^\epsilon\} = \cosh \beta H \sum_{\chi \in \mathbf{Z}_2^{add}} (-1)^{\chi^\epsilon} (\tanh \beta H)^{\tau((-1)^\chi)}$$

we get

$$\begin{aligned} \exp\{\beta H \sum_{s_i^p \in P(G(M))} (-1)^{\sigma^p(s_i^p)}\} &= (\cosh \beta H)^{\#(G(M);p)} \\ &\times \left(\sum_{\chi^p \in C^p(P(G(M)), \mathbf{Z}_2^{add})} (-1)^{\langle \sigma^p, \chi^p \rangle} (\tanh \beta H)^{|\chi^p|_{P(G(M))}} \right). \end{aligned} \quad (3.17)$$

The bilinear form $\langle \sigma^p, \chi^p \rangle$, the mapping $\tau((-1)^\chi)$ and the length $|\chi^p|_{P(G(M))}$ are given by the relations (2.4), (2.18) and (2.50). If the magnetic field $H(s_i^1)$ depends on the edge s_i^1 orientation, the right-hand side of the equality (3.17) is not so simple. The relations (2.16), (3.16), (3.17) imply

$$\begin{aligned} Z_p(J, H; G(M)) &= Z_p(J, 0; G(M)) (\cosh \beta H)^{\#(G(M);p)} S_p(J, H; G(M)), \\ S_p(J, H; G(M)) &= \sum_{\chi^p \in B_p(P(G(M)), \mathbf{Z}_2^{add})} (\tanh \beta H)^{|\chi^p|_{P(G(M))}} \alpha(\chi^p; G(M)). \end{aligned} \quad (3.18)$$

The correlation function (2.16) is equal to zero for $\chi^p \notin B_p(P(G(M)), \mathbf{Z}_2^{add})$. The relation (3.18) for $p = 0$ is obtained in the paper [9]. For the vacuum ($J(s_i^{p+1}) = 0$) the relations (2.10), (3.16) imply

$$Z_p(0, H; G(M)) = (2 \cosh \beta H)^{\#(G(M);p)}. \quad (3.19)$$

The "energy" of the constant magnetic field $H(s_j^p) = H$ for a non-boundary cell $s_i^{p+1} \in G(M)$ is the product H^{2p+2} of the magnetic fields corresponding to $2p+2$ boundary cells $s_j^p \in \partial s_i^{p+1}$. The total "energy" of the magnetic field $H(s_j^p) = H$ is the sum over the cells s_i^{p+1}

$$\sum_{s_i^{p+1} \in P(G(M))} \left(\prod_{s_j^p : (s_i^{p+1}, s_j^p) = 1} H(s_j^p) \right) \approx (\#(G(M); p+1)) H^{2p+2}. \quad (3.20)$$

We neglect the boundary cells s_i^{p+1} from $P(G(M))$. The "re-normalized" magnetic field

$$H(s_j^p) = (\#(G(M); p+1))^{-1/(2p+2)} \beta^{-1} \tanh \beta H \quad (3.21)$$

yields the constant "re-normalized total energy" (3.20). In view of the relations (3.18), (3.19) we get the spontaneous magnetization for the "re-normalized" magnetic field (3.21)

$$\lim_{G(M) \rightarrow \mathbf{Z} \times d} \frac{\partial}{\partial x} \left(\ln \frac{Z_p(J, (\#(G(M); p+1))^{-1/(2p+2)} \beta^{-1} x; G(M))}{Z_p(0, (\#(G(M); p+1))^{-1/(2p+2)} \beta^{-1} x; G(M))} \right)_{x = \tanh \beta H} =$$

$$\lim_{G(M) \rightarrow \mathbf{Z} \times d} \frac{\partial}{\partial x} \left(\ln S_p(J, (\#(G(M); p+1))^{-1/(2p+2)} \beta^{-1} x; G(M)) \right)_{x = \tanh \beta H}. \quad (3.22)$$

Let us introduce the set of the connected cochains $\lambda_{1i_1}^{p+1}(M) \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$:

$$\partial \lambda_{1i_1}^{p+1}(M) \neq 0;$$

$$(s_i^{p+1} : s_l^p)(s_j^{p+1} : s_l^p) = 0, \quad \lambda_{1i_1}^{p+1}(M)(s_i^{p+1}) = 1, \quad \lambda_{1j_1}^{p+1}(M)(s_j^{p+1}) = 1, \quad i_1 \neq j_1, \quad (3.23)$$

$i_1, j_1 = 1, \dots, k_1$. The integer $k_1 \leq \#(G(M); p+1)$. The relations (2.37), (3.18) imply

$$S_p(J, (\#(G(M); p+1))^{-1/(2p+2)} \beta^{-1} \tanh(\beta H); G(M)) =$$

$$1 + \sum_{1 \leq k_1} \sum_{\lambda_{1i_1}^{p+1}(M) : (3.23)} \left(\prod_{i_1=1}^{k_1} \|\lambda_{1i_1}^{p+1}(M)\|_{J, G(M)} \right)$$

$$\times \left(\tanh((\#(G(M); p+1))^{-1/(2p+2)} \tanh(\beta H)) \right)^{\sum_{i_1=1}^{k_1} |\partial \lambda_{1i_1}^{p+1}(M)|_{P(G(M))}}$$

$$\times \left(1 + \alpha \left(\sum_{i_1=1}^{k_1} \partial \lambda_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(M); G(M) \right) \right)^{-1}. \quad (3.24)$$

Due to the definition (2.13) the correlation function $\alpha(0; G(M)) = 1$. It is easy to verify

$$\frac{d}{dx} \tanh x = (\cosh x)^{-2}, \quad |\tanh x| \leq |x|,$$

$$x = (\#(G(M); p+1))^{-1/(2p+2)} \tanh(\beta H). \quad (3.25)$$

The inequality (2.53) implies

$$\left(1 + \alpha \left(\sum_{i_1=1}^{k_1} \partial \lambda_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \lambda_{1i_1}^{p+1}(M); G(M) \right) \right)^{-1} \leq 1. \quad (3.26)$$

If the support of the connected cochain $\mu^{p+1} \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$ consists of the only cell s_i^{p+1} , then the length of its boundary

$$|\partial \mu^{p+1}|_{P(G(M))} = 2p + 2. \quad (3.27)$$

If the connected cochain $\mu^{p+1} \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$ satisfies the equation (3.27), the length $|\mu^{p+1}|_{P(G(M))} = 1$ for $p = 1, d = 2$. The length of the cochain $|\mu^{p+1}|_{P(G(M))}$ satisfying the equation (3.27) may be practically arbitrary for $p = 0, d = 1, 2, 3$ and for $p = 1, d = 3$. Let us introduce the connected cochains $\mu_{1i_1}^{p+1}(M) \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$, $i_1 = 1, \dots, k_1$, satisfying the equations

$$|\partial \mu_{1i_1}^{p+1}(M)|_{P(G(M))} = 2p + 2, \quad i_1 = 1, \dots, k_1;$$

$$(s_i^{p+1} : s_l^p)(s_j^{p+1} : s_l^p) = 0, \mu_{1i_1}^{p+1}(M)(s_i^{p+1}) = 1, \mu_{1j_1}^{p+1}(M)(s_j^{p+1}) = 1, i_1 \neq j_1, \quad (3.28)$$

$i_1, j_1 = 1, \dots, k_1$.

The ratio of the total number of the shifts of the connected cochain $\lambda^{p+1}(M)$ from the group $C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$ in the graph $G(M)$ and of the number $\#(G(M); p+1)$ tends to one when $G(M) \rightarrow \mathbf{Z}^{\times d}$. Let the sign of the interaction energy $J(s_i^{p+1})$ be independent of the cell s_i^{p+1} and the interaction energy $J(s_i^{p+1})$ satisfy the inequality (2.56). By making use of the inequalities (2.55), (3.25), (3.26) it is possible to prove that in the right-hand side of the equality (3.24) the terms with the cochains (3.28) only may be nonzero when $G(M) \rightarrow \mathbf{Z}^{\times d}$

$$\begin{aligned} & \lim_{G(M) \rightarrow \mathbf{Z}^{\times d}} S_p(J, (\#(G(M); p+1))^{-1/(2p+2)} \beta^{-1} \tanh(\beta H); G(M)) = \\ & 1 + \lim_{G(M) \rightarrow \mathbf{Z}^{\times d}} \sum_{1 \leq k_1} \sum_{\mu_{1i_1}^{p+1}(M): (3.28)} \left(\prod_{i_1=1}^{k_1} \|\mu_{1i_1}^{p+1}(M)\|_{J, G(M)} \right) \\ & \times \left(\tanh((\#(G(M); p+1))^{-1/(2p+2)} \tanh(\beta H)) \right)^{2k_1(p+1)} \\ & \times \left(1 + \alpha \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M); G(M) \right) \right)^{-1}. \end{aligned} \quad (3.29)$$

Due to the relations (2.20), (2.21), (2.23), (2.24) the left-hand side of the inequality (3.26) is equal to one for $p = 0, d = 1$ and $p = 1, d = 2$. For these theories the proof of the relation (3.52) similar to (3.15) is continued from the relation (3.50). The correlation function

$$\alpha \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M); G(M) \right)$$

in the right-hand side of the equality (3.29) satisfies the equation (2.43). The connected cochains $\lambda_{2i_2}^{p+1}(M)$, $i_2 = 1, \dots, k_2$, in the equality (2.43) satisfy the equations (2.40) for the cochains $\mu_{1i_1}^{p+1}(M)$ instead of the cochains $\lambda_{1i_1}^{p+1}(M)$. We divide the cochains $\lambda_{2i_2}^{p+1}(M)$, $i_2 = 1, \dots, k_2$, into two sets. The first set consists of the connected cochains $\mu_{2i_2}^{p+1}(M) \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$, $i_2 = 1, \dots, k_2$, satisfying the equations: for every number $i_2 = 1, \dots, k_2$ there is only one number $i_1 = 1, \dots, k_1$ such that

$$\exists i, j, l, (s_i^{p+1} : s_l^p)(s_j^{p+1} : s_l^p) = 1, \mu_{2i_2}^{p+1}(M)(s_i^{p+1}) = 1, \mu_{1i_1}^{p+1}(M)(s_j^{p+1}) = 1. \quad (3.30)$$

The connected cochains $\mu_{2i_2}^{p+1}(M) \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$, $i_2 = 1, \dots, k_2$, satisfy also the equations similar to the first and the third equations (2.40)

$$\partial \mu_{2i_2}^{p+1}(M) = 0;$$

$$(s_i^{p+1} : s_l^p)(s_j^{p+1} : s_l^p) = 0, \mu_{2i_2}^{p+1}(M)(s_i^{p+1}) = 1, \mu_{2j_2}^{p+1}(M)(s_j^{p+1}) = 1, i_2 \neq j_2, \quad (3.31)$$

$i_2, j_2 = 1, \dots, k_2$.

The second set consists of the connected cochains $\nu_{2i_2}^{p+1}(M) \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$, $i_2 = 1, \dots, k_2$ satisfying the equations: for every number $i_2 = 1, \dots, k_2$ there exists the number $i_1 = 1, \dots, k_1$ such that

$$\exists i, j, l, (s_i^{p+1} : s_l^p)(s_j^{p+1} : s_l^p) = 1, \nu_{2i_2}^{p+1}(M)(s_i^{p+1}) = 1, \mu_{1i_1}^{p+1}(M)(s_j^{p+1}) = 1 \quad (3.32)$$

and there exists the number $i_2 = 1, \dots, k_2$ such that

$$\begin{aligned} \exists i, j, l, (s_i^{p+1} : s_l^p)(s_j^{p+1} : s_l^p) = 1, \nu_{2i_2}^{p+1}(M)(s_i^{p+1}) = 1, \mu_{1i_1}^{p+1}(M)(s_j^{p+1}) = 1, \\ \exists i, j, l, (s_i^{p+1} : s_l^p)(s_j^{p+1} : s_l^p) = 1, \nu_{2i_2}^{p+1}(M)(s_i^{p+1}) = 1, \mu_{1j_1}^{p+1}(M)(s_j^{p+1}) = 1 \end{aligned} \quad (3.33)$$

for at least two different numbers $i_1, j_1 = 1, \dots, k_1$. The connected cochains $\nu_{2i_2}^{p+1}(M) \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$, $i_2 = 1, \dots, k_2$, satisfy also the equations similar to the equations (3.31)

$$\partial \nu_{2i_2}^{p+1}(M) = 0;$$

$$(s_i^{p+1} : s_l^p)(s_j^{p+1} : s_l^p) = 0, \nu_{2i_2}^{p+1}(M)(s_i^{p+1}) = 1, \nu_{2j_2}^{p+1}(M)(s_j^{p+1}) = 1, i_2 \neq j_2, \quad (3.34)$$

$i_2, j_2 = 1, \dots, k_2$. We divide the sum (2.43) into two parts

$$\begin{aligned} \alpha_2 \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M); G(M) \right) = \\ \sum_{1 \leq k_2} \sum_{\mu_{2i_2}^{p+1}(M): (3.30), (3.31)} \left(\prod_{i_2=1}^{k_2} \|\mu_{2i_2}^{p+1}(M)\|_{J, G(M)} \right) \\ \times \left(1 + \alpha \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M), \sum_{i_2=1}^{k_2} \mu_{2i_2}^{p+1}(M); G(M) \right) \right)^{-1}, \end{aligned} \quad (3.35)$$

$$\begin{aligned} \beta_2 \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M); G(M) \right) = \\ \sum_{1 \leq k_2} \sum_{\nu_{2i_2}^{p+1}(M): (3.32)-(3.34)} \left(\prod_{i_2=1}^{k_2} \|\nu_{2i_2}^{p+1}(M)\|_{J, G(M)} \right) \\ \times \left(1 + \alpha \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M), \sum_{i_2=1}^{k_2} \nu_{2i_2}^{p+1}(M); G(M) \right) \right)^{-1}. \end{aligned} \quad (3.36)$$

If the sign of the interaction energy $J(s_i^{p+1})$ is independent of the cell s_i^{p+1} and the interaction energy $J(s_i^{p+1})$ satisfies the inequality (2.56), then the inequalities (2.52), (2.53), (2.55) and the equalities (3.35), (3.36) imply

$$\begin{aligned} 0 \leq \alpha_2 \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M); G(M) \right) < 1, \\ 0 \leq \beta_2 \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M); G(M) \right) < 1, \end{aligned} \quad (3.37)$$

$$\begin{aligned} \left(1 + \alpha \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M); G(M) \right) \right)^{-1} = \\ \sum_{m=0}^{\infty} \left(1 + \alpha_2 \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M); G(M) \right) \right)^{-m-1} \\ \times \left(-\beta_2 \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M); G(M) \right) \right)^m. \end{aligned} \quad (3.38)$$

Let us substitute the equality (3.38) into the right-hand side of the equality (3.29). Now every term of the sum (3.29) with the term of the sum (3.38) for $m \geq 1$ contains the cochain $\nu_{2i_2}^{p+1}(M)$ connecting together due to the relations (3.33) at least two cochains $\mu_{1j_1}^{p+1}(M)$, $\mu_{1l_1}^{p+1}(M)$ in the sum (3.29). These connected together cochains can move on the graph $G(M)$ as one connected cochain. Hence the inequality (3.25) implies

$$\begin{aligned} \lim_{G(M) \rightarrow \mathbf{Z}^{\times d}} S_p(J, (\#(G(M); p+1))^{-1/(2p+2)} \beta^{-1} \tanh(\beta H); G(M)) = \\ 1 + \lim_{G(M) \rightarrow \mathbf{Z}^{\times d}} \sum_{1 \leq k_1} \sum_{\mu_{1i_1}^{p+1}(M): (3.28)} \left(\prod_{i_1=1}^{k_1} \|\mu_{1i_1}^{p+1}(M)\|_{J, G(M)} \right) \\ \times \left(\tanh((\#(G(M); p+1))^{-1/(2p+2)} \tanh(\beta H)) \right)^{2k_1(p+1)} \\ \times \left(1 + \alpha_2 \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M); G(M) \right) \right)^{-1}. \end{aligned} \quad (3.39)$$

We define the set of the cochains $\mu_{ni_n}^{p+1}(M)$, $i_n = 1, \dots, k_n$, $n = 1, 2, \dots$, from the group $C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$. The cochains $\mu_{1i_1}^{p+1}(M)$, $i_1 = 1, \dots, k_1$, satisfy the equations (3.28). The cochains $\mu_{ni_n}^{p+1}(M)$, $i_n = 1, \dots, k_n$, for $n \geq 2$ satisfy the equations: for every number i_n there exists the sequence of the cochains $\mu_{1i_1}^{p+1}(M), \dots, \mu_{ni_n}^{p+1}(M)$ such that

$$\exists i, j, l, (s_i^{p+1} : s_l^p)(s_j^{p+1} : s_l^p) = 1, \mu_{mi_m}^{p+1}(M)(s_i^{p+1}) = 1, \mu_{(m+1)i_{m+1}}^{p+1}(M)(s_j^{p+1}) = 1, \quad (3.40)$$

$m = 1, \dots, n-1$, and any two sequences (3.40)

$$\exists i, j, l, (s_i^{p+1} : s_l^p)(s_j^{p+1} : s_l^p) = 1, \mu_{mi_m}^{p+1}(M)(s_i^{p+1}) = 1, \mu_{(m+1)i_{m+1}}^{p+1}(M)(s_j^{p+1}) = 1,$$

$$\exists i, j, l, (s_i^{p+1} : s_l^p)(s_j^{p+1} : s_l^p) = 1, \mu_{mj_m}^{p+1}(M)(s_i^{p+1}) = 1, \mu_{(m+1)j_{m+1}}^{p+1}(M)(s_j^{p+1}) = 1, \quad (3.41)$$

$m = 1, \dots, n-1$, with the same end: $i_n = j_n$ have the same beginning: $i_1 = j_1$. The cochains $\mu_{ni_n}^{p+1}(M)$, $i_n = 1, \dots, k_n$, for $n \geq 2$ satisfy also the equations similar to the equations (2.45)

$$\partial \mu_{ni_n}^{p+1}(M) = 0, \quad i_n = 1, \dots, k_n;$$

$$(s_i^{p+1} : s_l^p)(s_j^{p+1} : s_l^p) = 0, \quad \mu_{mi_m}^{p+1}(M)(s_i^{p+1}) = 1, \quad \mu_{ni_n}^{p+1}(M)(s_j^{p+1}) = 1,$$

$$i_m = 1, \dots, k_m, \quad i_n = 1, \dots, k_n, \quad m = 1, \dots, n-2 \geq 1;$$

$$(s_i^{p+1} : s_l^p)(s_j^{p+1} : s_l^p) = 0, \quad \mu_{ni_n}^{p+1}(M)(s_i^{p+1}) = 1, \quad \mu_{nj_n}^{p+1}(M)(s_j^{p+1}) = 1, \quad i_n \neq j_n, \quad (3.42)$$

$i_n, j_n = 1, \dots, k_n$. For $n = 2$ the equations (3.40) - (3.42) coincide with the equations (3.30), (3.31). The equations (3.41) mean that the set of the cochains $\mu_{li_i}^{p+1}(M) \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$, $i_l = 1, \dots, k_l$, $l = 1, 2, \dots$, is k_1 cochain trees with the trunks $\mu_{11}^{p+1}(M), \dots, \mu_{1k_1}^{p+1}(M)$. By repeating the proof of the equality (3.39) it is possible to prove that the correlation function

$$\alpha_2 \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M); G(M) \right)$$

in the right-hand side of the equality (3.39) may be considered as the first term of the sequence of the correlation functions

$$\begin{aligned} \alpha_n \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M), \dots, \sum_{i_{n-1}=1}^{k_{n-1}} \mu_{(n-1)i_{n-1}}^{p+1}(M); G(M) \right) = \\ \sum_{1 \leq k_n} \sum_{\mu_{ni_n}^{p+1}(M): (3.40)-(3.42)} \left(\prod_{i_n=1}^{k_n} \|\mu_{ni_n}^{p+1}(M)\|_{J,G(M)} \right) \\ \times \left(1 + \alpha_{n+1} \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M), \dots, \sum_{i_n=1}^{k_n} \mu_{ni_n}^{p+1}(M); G(M) \right) \right)^{-1}, \end{aligned} \quad (3.43)$$

$n = 2, \dots, N-1$. For $n = N$ the correlation function

$$\begin{aligned} \alpha_N \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M), \dots, \sum_{i_{N-1}=1}^{k_{N-1}} \mu_{(N-1)i_{N-1}}^{p+1}(M); G(M) \right) = \\ \sum_{1 \leq k_N} \sum_{\mu_{Ni_N}^{p+1}(M): (3.40)-(3.42), n \rightarrow N} \left(\prod_{i_N=1}^{k_N} \|\mu_{Ni_N}^{p+1}(M)\|_{J,G(M)} \right) \\ \times \left(1 + \alpha \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M), \dots, \sum_{i_N=1}^{k_N} \mu_{Ni_N}^{p+1}(M); G(M) \right) \right)^{-1}. \end{aligned} \quad (3.44)$$

N is an arbitrary integer independent of the graph $G(M)$. For $N = 2$ the relation (3.44) coincides with the relation (3.35).

Let us define the sequence of the correlation functions

$$\alpha_n^{(N)} \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M), \dots, \sum_{i_{n-1}=1}^{k_{n-1}} \mu_{(n-1)i_{n-1}}^{p+1}(M); G(M) \right),$$

$n = 2, \dots, N-1$, satisfying the relations (3.43) where the correlation function

$$\alpha_N^{(N)} \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M), \dots, \sum_{i_{N-1}=1}^{k_{N-1}} \mu_{(N-1)i_{N-1}}^{p+1}(M); G(M) \right) = 0 \quad (3.45)$$

instead of the correlation function (3.44). The relations (3.43) - (3.45) imply

$$\begin{aligned} \alpha_n^{(N)} \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M), \dots, \sum_{i_{n-1}=1}^{k_{n-1}} \mu_{(n-1)i_{n-1}}^{p+1}(M); G(M) \right) \\ - \alpha_n \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M), \dots, \sum_{i_{n-1}=1}^{k_{n-1}} \mu_{(n-1)i_{n-1}}^{p+1}(M); G(M) \right) = \\ \sum_{1 \leq k_n} \sum_{\mu_{ni_n}^{p+1}(M): (3.40)-(3.42)} \left(\prod_{i_n=1}^{k_n} \|\mu_{ni_n}^{p+1}(M)\|_{J,G(M)} \right) \\ \times \left(1 + \alpha_{n+1} \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M), \dots, \sum_{i_n=1}^{k_n} \mu_{ni_n}^{p+1}(M); G(M) \right) \right)^{-1} \end{aligned}$$

$$\begin{aligned}
& \times \left(1 + \alpha_{n+1}^N \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M), \dots, \sum_{i_n=1}^{k_n} \mu_{ni}^{p+1}(M); G(M) \right) \right)^{-1} \\
& \times \left(\alpha_{n+1} \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M), \dots, \sum_{i_n=1}^{k_n} \mu_{ni}^{p+1}(M); G(M) \right) \right. \\
& \left. - \alpha_{n+1}^{(N)} \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M), \dots, \sum_{i_n=1}^{k_n} \mu_{ni}^{p+1}(M); G(M) \right) \right), \quad (3.46)
\end{aligned}$$

$$\begin{aligned}
& \alpha_N^{(N)} \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M), \dots, \sum_{i_{N-1}=1}^{k_{N-1}} \mu_{(N-1)i_{N-1}}^{p+1}(M); G(M) \right) \\
& - \alpha_N \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M), \dots, \sum_{i_{N-1}=1}^{k_{N-1}} \mu_{(N-1)i_{N-1}}^{p+1}(M); G(M) \right) = \\
& - \alpha_N \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M), \dots, \sum_{i_{N-1}=1}^{k_{N-1}} \mu_{(N-1)i_{N-1}}^{p+1}(M); G(M) \right). \quad (3.47)
\end{aligned}$$

If the sign of the interaction energy $J(s_i^{p+1})$ is independent of the cell s_i^{p+1} , then the inequality (2.52) and the definitions (3.43) - (3.45) imply

$$\begin{aligned}
& \alpha_n^{(N)} \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M), \dots, \sum_{i_{n-1}=1}^{k_{n-1}} \mu_{(n-1)i_{n-1}}^{p+1}(M); G(M) \right) \geq 0, \\
& \alpha_n \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M), \dots, \sum_{i_{n-1}=1}^{k_{n-1}} \mu_{(n-1)i_{n-1}}^{p+1}(M); G(M) \right) \geq 0, \quad (3.48)
\end{aligned}$$

$n = 2, \dots, N$. If the sign of the interaction energy $J(s_i^{p+1})$ is independent of the cell s_i^{p+1} and the interaction energy $J(s_i^{p+1})$ satisfies the inequality (2.56), then the inequalities (2.53), (2.55), (3.48) and the equalities (3.46), (3.47) imply that the difference (3.46), $n = 2$ is small for the large numbers N . Hence we get

$$\begin{aligned}
& \lim_{G(M) \rightarrow \mathbf{Z}^{\times d}} S_p(J, (\#(G(M); p+1))^{-1/(2p+2)} \beta^{-1} \tanh(\beta H); G(M)) = \\
& 1 + \lim_{N \rightarrow \infty} \lim_{G(M) \rightarrow \mathbf{Z}^{\times d}} \sum_{1 \leq k_1} (\tanh(\beta H))^{2k_1(p+1)} (\#(G(M); p+1))^{-k_1} \sum_{\mu_{1i}^{p+1}(M): (3.28)} \\
& \left\| \sum_{i=1}^{k_1} \mu_{1i}^{p+1}(M) \right\|_{J, G(M)} \left(1 + \alpha_2^{(N)} \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M); G(M) \right) \right)^{-1}. \quad (3.49)
\end{aligned}$$

By making use of the proof of the equality (3.49) we get

$$\begin{aligned}
& \lim_{G(M) \rightarrow \mathbf{Z}^{\times d}} ((k_1)!)^{-1} (\tanh(\beta H))^{2k_1(p+1)} (\#(G(M); p+1))^{-k_1} \\
& \times \left(\sum_{\chi^p \in B_p(P(G(M)), \mathbf{Z}_2^{odd}), |\chi^p|_{P(G(M))} = 2p+2} \alpha(\chi^p; G(M)) \right)^{k_1} = \lim_{N \rightarrow \infty} \lim_{G(M) \rightarrow \mathbf{Z}^{\times d}}
\end{aligned}$$

$$\begin{aligned}
& (\tanh(\beta H))^{2k_1(p+1)} (\#(G(M); p+1))^{-k_1} \sum_{\mu_{1i_1}^{p+1}(M): (3.28)} \left\| \sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1}(M) \right\|_{J, G(M)} \\
& \times \left(1 + \alpha_2^{(N)} \left(\sum_{i_1=1}^{k_1} \partial \mu_{1i_1}^{p+1}(M), \sum_{i_1=1}^k \mu_{1i_1}^{p+1}(M); G(M) \right) \right)^{-1}, \quad (3.50)
\end{aligned}$$

$k_1 = 1, 2, \dots$. All $(k_1)!$ possible ordering of the different cochains $\mu_{1i_1}^{p+1} \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add})$, $i = 1, \dots, k_1$, give the same sum

$$\sum_{i_1=1}^{k_1} \mu_{1i_1}^{p+1} \in C^{p+1}(P(G(M)), \mathbf{Z}_2^{add}).$$

It explains the multiplier $((k_1)!)^{-1}$ in the left-hand side of the equality (3.50). By making use of the equality (3.49) and summing up the equalities (3.50) we get

$$\begin{aligned}
& \lim_{G(M) \rightarrow \mathbf{Z}^{\times d}} S_p(J, (\#(G(M); p+1))^{-1/(2p+2)} \beta^{-1} \tanh(\beta H); G(M)) = \\
& \lim_{G(M) \rightarrow \mathbf{Z}^{\times d}} \exp \left\{ (\tanh(\beta H))^{2p+2} (\#(G(M); p+1))^{-1} \times \right. \\
& \left. \sum_{\chi^p \in B_p(P(G(M)), \mathbf{Z}_2^{add}), |\chi^p|_{P(G(M))} = 2p+2} \alpha(\chi^p; G(M)) \right\}. \quad (3.51)
\end{aligned}$$

By making use of the equality (3.22) and of the proof of the equality (3.51) we can prove

$$\begin{aligned}
& \lim_{G(M) \rightarrow \mathbf{Z}^{\times d}} \frac{\partial}{\partial x} \left(\ln \frac{Z_p(J, (\#(G(M); p+1))^{-1/(2p+2)} \beta^{-1} x; G(M))}{Z_p(0, (\#(G(M); p+1))^{-1/(2p+2)} \beta^{-1} x; G(M))} \right)_{x = \tanh \beta H} = \\
& 2(p+1)(\tanh \beta H)^{2p+1} \times \\
& \lim_{G(M) \rightarrow \mathbf{Z}^{\times d}} (\#(G(M); p+1))^{-1} \sum_{\chi^p \in B_p(P(G(M)), \mathbf{Z}_2^{add}), |\chi^p|_{P(G(M))} = 2p+2} \alpha(\chi^p; G(M)). \quad (3.52)
\end{aligned}$$

The equality (3.52) is proved for $p = 0$, $d = 1, 2, 3$ and for $p = 1$, $d = 2, 3$. The equality (3.52) for $p = 0$, $d = 1$ and the constant interaction energy $J(s_j^1)$ coincides with the equality (3.15). The equality (3.52) for $p = 0$, $d = 2$ is proved in the paper [9].

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